The interplay between calculation and reasoning

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Introduction

To what extent can we automate assessment of steps in students' working?

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To what extent can we automate assessment of steps in students' working?

- Today

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To what extent can we automate assessment of steps in students' working?

- Today
- Tomorrow

Introduction

To what extent can we automate assessment of steps in students' working?

- Today
- Tomorrow
- Ever...

Current STACK interface

Tidy question | Question tests & deployed versions

Prove by induction that

$$\sum_{k=1}^{n} k^2 = \frac{1}{6} n(n+1)(2n+1).$$

```
"Set P(n) to be the statement that"
sum(k^2,k,1,n) = (n*(n+1)*(2*n+1))/6
"Then P(1) is the statement"
1^2 = 1*(1+1)*(2*1+1)/6
1 = 1
"So P(1) holds.  Now assume P(n) is true."
sum(k^2,k,1,n) = (n*(n+1)*(2*n+1))/6
sum(k^2,k,1,n) + (n+1)^2= (n*(n+1)*(2*n+1))/6+(n+1)^2
sum(k^2,k,1,n+1) = ((n+1)*(n*(2*n+1)+6*(n+1)))/6
sum(k^2,k,1,n+1) = ((n+1)*(2*n^2+7*n+6))/6
sum(k^2,k,1,n+1) = ((n+1)*(n+2)*(2*(n+1)+1))/6
```

Set P(n) to be the statement that

$$\sum_{k=1}^{n} k^2 = \frac{n\cdot(n+1)\cdot(2\cdot n+1)}{6}$$

Then P(1) is the statement

$$1^2 = \frac{1\cdot(1+1)\cdot(2\cdot 1+1)}{6}$$

$$\Leftrightarrow \quad 1 = 1$$

So P(1) holds. Now assume P(n) is true.

$$\sum_{k=1}^{n} k^2 = \frac{n\cdot(n+1)\cdot(2\cdot n+1)}{6}$$

$$\Leftrightarrow \quad \sum_{k=1}^{n} k^2 + (n+1)^2 = \frac{n\cdot(n+1)\cdot(2\cdot n+1)}{6} + (n+1)^2$$

$$\Leftrightarrow \quad \sum_{k=1}^{n+1} k^2 = \frac{(n+1)\cdot(n\cdot(2\cdot n+1)+6\cdot(n+1))}{6}$$

$$\Leftrightarrow \quad \sum_{k=1}^{n+1} k^2 = \frac{(n+1)\cdot\left(2\cdot n^2+7\cdot n+6\right)}{6}$$

$$\Leftrightarrow \quad \sum_{k=1}^{n+1} k^2 = \frac{(n+1)\cdot(n+2)\cdot(2\cdot(n+1)+1)}{6}$$

Calculation and Reasoning

Calculation: “a deliberate process that transforms one or more inputs into one or more results" (Wikipedia)

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By definition: we must perform a calculation in automatic assessment.

What forms of reasoning can be reduced to a calculation?

Reasoning by equivalence

Work line by line: adjacent lines are “equivalent".

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$$
\begin{aligned}
& \log_3(x+17)-2 && = \log_3(2x) \quad (x>0, x>-17) \\
\Leftrightarrow & \log_3(x+17)-\log_3(2x) && = 2 \\
\Leftrightarrow & \log_3\left(\frac{x+17}{2x}\right) && = 2 \\
\Leftrightarrow & \frac{x+17}{2x} && = 3^2 = 9 \\
\Leftrightarrow & x+17 && = 18x \\
\Leftrightarrow & x && = 1.
\end{aligned}
$$

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The above is a *single mathematical entity: the argument.*

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\end{aligned}$$

The above is a *single mathematical entity: the argument.* (For Christian, *et al.*)

The above is a single (long) *English sentence.*

Importance of RE in undergraduate mathematics

Reasoning by equivalence is important for the following reasons.

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3. Included in many methods, e.g. solving ODEs.

Importance of RE in undergraduate mathematics

Reasoning by equivalence is important for the following reasons.

1. Start of proof & rigour (deductive geometry?)
2. Contains logic and extended calculation
3. Included in many methods, e.g. solving ODEs.
4. Key part of many pure mathematics proofs
    - Induction step
    - ϵ-δ proofs.

Importance of RE in school mathematics

Reasoning by equivalence is the primary form of reasoning.

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Reasoning by equivalence is the primary form of reasoning.
1/3 of marks in the IB exams are awarded for RE.

Reasoning by equivalence has a long history

A “universal scientific language" would enable us to

> *judge immediately whether propositions presented to us are proved ... with the guidance of symbols alone, by a sure truly analytical method.*

Boole *Laws of thought* 1854

"to go under, over, and beyond" Aristotle's logic.

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Mathematical foundations involving equations.

Pell's *Algebra* 1668

Prob. IX. There be three Numbers in *continual* Proportion; their sum is 74, and the sum of their Squares 1924.

$a = ?$	1	$a+b+c=74$
$b = ?$	2	$aa+bb+cc=1924$
$c = ?$	3	$a.b::b.c$
1 ⊕ 2	4	$aa+bb+cc+2ab+2ac+2bc=5476$
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10 ⊕ 2	12	$aa+2ac+cc=2500$
11 ✱ 4	13	$4ac=2304$
12 — 13	14	$aa-2ac+cc=196$
14 ω 2	15	$a-c = \pm 14$
10 + 15	16	$2a=64$ Also $=36$
16 ÷ 2	17	$A=32$ Also $=18$
10 — 17	18	$C=18$ Also $=32$

Equivalence reasoning and STACK

Goal: develop STACK to assess reasoning by equivalence.

Equivalence reasoning

Applies to *equations*.

$$\begin{array}{ll} & (x-5)^2 - 16 = 0 \\ \Leftrightarrow & (x-5)^2 = 16 \\ \Leftrightarrow & x - 5 = \pm(4) \\ \Leftrightarrow & x - 5 = 4 \text{ or } x - 5 = -4 \\ \Leftrightarrow & x = 1 \text{ or } x = 9 \end{array}$$

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Equivalence class of expressions defined by the *solution set*.

Solving an equation

Solving is

- progressive transformations;
- representatives of the class;
- ending in a certain form.

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- representatives of the class;
- ending in a certain form.

E.g. polynomial equation $\rightarrow x =?$ or $x =? \cdots$

Design decisions: repeated roots?

$$
\begin{array}{ll}
 & x^2 - 6 \cdot x = -9 \\
\Leftrightarrow & (x-3)^2 = 0 \\
\text{(Same roots)} & x - 3 = 0 \\
\Leftrightarrow & x = 3
\end{array}
$$

Design decisions: which field?

$\mathbb{R}$ or $\mathbb{C}$?

$$
\begin{array}{ll}
 & x^3 - 1 = 0 \\
\Leftrightarrow & (x-1)\cdot\left(x^2+x+1\right) = 0 \\
? & x = 1
\end{array}
$$

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\Leftrightarrow & (x-1)\cdot\left(x^2+x+1\right) = 0 \\
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\end{array}
$$

STACK currently works over $\mathbb{C}$.

$$
\begin{array}{rl}
 & x^3 - 1 = 0 \\
\Leftrightarrow & (x-1)\cdot\left(x^2+x+1\right) = 0 \\
\Leftrightarrow & x = 1 \text{ or } x^2+x+1 = 0 \\
\Leftrightarrow & x = 1 \text{ or } x = \frac{-\left(\sqrt{3}\cdot i+1\right)}{2} \text{ or } x = \frac{\sqrt{3}\cdot i-1}{2}
\end{array}
$$

Equating expressions

- Similar to equivalence reasoning.
- Expressions, (not equations).

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$$
\begin{aligned}
& 2\cdot\left(a^2\cdot b^2+b^2\cdot c^2+c^2\cdot a^2\right)-\left(a^4+b^4+c^4\right)\\
=\ & 4\cdot a^2\cdot b^2-\left(a^4+b^4+c^4+2\cdot a^2\cdot b^2-2\cdot b^2\cdot c^2-2\cdot c^2\cdot a^2\right)\\
=\ & (2\cdot a\cdot b)^2-\left(b^2+a^2-c^2\right)^2\\
=\ & \left(2\cdot a\cdot b+b^2+a^2-c^2\right)\cdot\left(2\cdot a\cdot b-b^2-a^2+c^2\right)\\
=\ & \left((a+b)^2-c^2\right)\cdot\left(c^2-(a-b)^2\right)\\
=\ & (a+b+c)\cdot(a+b-c)\cdot(c+a-b)\cdot(c-a+b)
\end{aligned}
$$

Equating expressions

- Similar to equivalence reasoning.
- Expressions, (not equations).

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& 2 \cdot \left(a^2 \cdot b^2 + b^2 \cdot c^2 + c^2 \cdot a^2\right) - \left(a^4 + b^4 + c^4\right) \\
= {} & 4 \cdot a^2 \cdot b^2 - \left(a^4 + b^4 + c^4 + 2 \cdot a^2 \cdot b^2 - 2 \cdot b^2 \cdot c^2 - 2 \cdot c^2 \cdot a^2\right) \\
= {} & (2 \cdot a \cdot b)^2 - \left(b^2 + a^2 - c^2\right)^2 \\
= {} & \left(2 \cdot a \cdot b + b^2 + a^2 - c^2\right) \cdot \left(2 \cdot a \cdot b - b^2 - a^2 + c^2\right) \\
= {} & \left((a+b)^2 - c^2\right) \cdot \left(c^2 - (a-b)^2\right) \\
= {} & (a+b+c) \cdot (a+b-c) \cdot (c+a-b) \cdot (c-a+b)
\end{aligned}
$$

Hidden quantifiers: for all values of all variables.

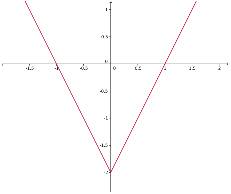
Equating expressions vs equivalence reasoning.

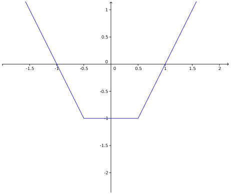
$$|x - 1/2| + |x + 1/2| = 2.$$

$$\Leftrightarrow |x| = 1$$

But

$$|x| - 1 \neq |x - 1/2| + |x + 1/2| = 2.$$





Equivalence classes vs explicit steps

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$$\begin{array}{rl} & (x+3)\cdot(2-x)=4 \\ \Leftrightarrow & x+3=4 \text{ or } 2-x=4 \\ \Leftrightarrow & x=1 \text{ or } x=-2 \end{array}$$

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Two options for the architecture:

- Membership of an equivalence class.
- Sequence of legitimate steps.

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Two options for the architecture:

- Membership of an equivalence class.
- Sequence of legitimate steps.

(Good nonsense is surprisingly hard to find.....)

Implication vs equivalence

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\begin{aligned}
& a = b \\
\Rightarrow\quad & a^2 = b^2
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E.g.

$$\begin{aligned} & \sqrt{3 \cdot x + 4} = 2 + \sqrt{x+2} \\ \Rightarrow \quad & 3 \cdot x + 4 = 4 + 4 \cdot \sqrt{x+2} + (x+2) \\ \Leftrightarrow \quad & x - 1 = 2 \cdot \sqrt{x+2} \\ \Rightarrow \quad & x^2 - 2 \cdot x + 1 = 4 \cdot x + 8 \\ \Leftrightarrow \quad & x^2 - 6 \cdot x - 7 = 0 \\ \Leftrightarrow \quad & (x-7) \cdot (x+1) = 0 \\ \Leftrightarrow \quad & x = 7 \text{ or } x = -1 \end{aligned}$$

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E.g.

$$\begin{array}{rl} & \sqrt{3\cdot x+4} = 2+\sqrt{x+2} \\ \Rightarrow & 3\cdot x+4 = 4+4\cdot\sqrt{x+2}+(x+2) \\ \Leftrightarrow & x-1 = 2\cdot\sqrt{x+2} \\ \Rightarrow & x^2-2\cdot x+1 = 4\cdot x+8 \\ \Leftrightarrow & x^2-6\cdot x-7 = 0 \\ \Leftrightarrow & (x-7)\cdot(x+1) = 0 \\ \Leftrightarrow & x = 7 \text{ or } x = -1 \end{array}$$

1. These problems are out of fashion. (SHAME!)

Implication vs equivalence

$$\begin{array}{rl} & a = b \\ \Rightarrow & a^2 = b^2 \end{array}$$

E.g.

$$\begin{array}{rl} & \sqrt{3 \cdot x + 4} = 2 + \sqrt{x + 2} \\ \Rightarrow & 3 \cdot x + 4 = 4 + 4 \cdot \sqrt{x + 2} + (x + 2) \\ \Leftrightarrow & x - 1 = 2 \cdot \sqrt{x + 2} \\ \Rightarrow & x^2 - 2 \cdot x + 1 = 4 \cdot x + 8 \\ \Leftrightarrow & x^2 - 6 \cdot x - 7 = 0 \\ \Leftrightarrow & (x - 7) \cdot (x + 1) = 0 \\ \Leftrightarrow & x = 7 \text{ or } x = -1 \end{array}$$

1. These problems are out of fashion. (SHAME!)
2. Start with equivalence, and progressively add rules for feedback.

Rational expressions: role of domains?

$$
\begin{array}{ll}
 & \frac{x^2-4}{x-2} = 0 \\
? & x^2 - 4 = 0 \\
\Leftrightarrow & (x-2)\cdot(x+2) = 0 \\
\Leftrightarrow & x = -2 \text{ or } x = 2
\end{array}
$$

Rational expressions: role of domains?

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$$

Instead

$$
\begin{array}{rl}
 & \frac{x^2-4}{x-2} = 0 \\
\Leftrightarrow & \frac{(x-2)\cdot(x+2)}{x-2} = 0 \\
\Leftrightarrow & x + 2 = 0 \\
\Leftrightarrow & x = -2
\end{array}
$$

STACK and RE

Working

- Polynomials
- Rational expressions
- $\pm$
- $\surd$

STACK and RE

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- $\sqrt{}$

Future

- $|x|$
- Simultaneous equations
- Systems of inequalities

STACK and RE

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- Polynomials
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Future

- $|x|$
- Simultaneous equations
- Systems of inequalities

Distant future

- Trig

Students and RE

$$\text{Question 1: solve } \frac{x+5}{x-7} - 5 = \frac{4x-40}{13-x}.$$

$$\text{Question 2: solve } \sqrt{3x+4} = 2 + \sqrt{x+2}.$$

(147 participants: amongst highest achieving students in their generation)

Students and RE

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Outline results Q1:

- 9.5% of students showed any evidence of logical connectives
- 2 students checked their answer
- 1 student explicitly considered domains of definition, e.g. $x \neq 7$

Students and RE

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Outline results Q1:

- 9.5% of students showed any evidence of logical connectives
- 2 students checked their answer
- 1 student explicitly considered domains of definition, e.g. $x \neq 7$

Outline results Q2:

- 60% of students “finished” this problem getting $x = 7$, $x = -1$
- 16% checked and eliminated one solution
- 4 students showed any evidence of checking domains
- 3 students used any logical connectives

Teachers moaning about students....

There are few parts of algebra more important than the logic of the derivation of equations, and few, unhappily, that are treated in more slovenly fashion in elementary teaching.
Chrystal *(1893)*

CAS and RE

Current worksheet interfaces to CAS mimic students' approaches.

Solve the following equation.

```
(%i2) (x+5)/(x-7)-5 = (4*x-40)/(13-x);
```

(%o2) $\frac{x+5}{x-7}-5=\frac{4\,x-40}{13-x}$

```
(%i3) ratsimp(%);
```

(%o3) $-\frac{4\,x-40}{x-7}=-\frac{4\,x-40}{x-13}$

```
(%i4) %*(x-7)*(x-13);
```

(%o4) $-(x-13)\,(4\,x-40)=-(x-7)\,(4\,x-40)$

```
(%i5) %/(4*x-40);
```

(%o5) $13-x=7-x$

```
(%i6) %+x;
```

(%o6) $13=7$

Algebra and RE

To what extent do I want to automate current practice?

Algebra and RE

To what extent do I want to automate current practice?
What are the alternatives?

(Back 2010): “Structured derivations”

Find the values of *a* for which $-x^2 + ax + a - 3 < 0$ holds for all x.

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Find the values of *a* for which $-x^2 + ax + a - 3 < 0$ holds for all x.

$$\begin{array}{rc} & -x^2 + a \cdot x + a - 3 < 0 \\ \Leftrightarrow & a - 3 < x^2 - a \cdot x \\ \Leftrightarrow & a - 3 < \left(x - \frac{a}{2}\right)^2 - \frac{a^2}{4} \\ \Leftrightarrow & \frac{a^2}{4} + a - 3 < \left(x - \frac{a}{2}\right)^2 \end{array}$$

This inequality is required to be true for all x; it must be true when the right hand side takes its minimum value. This happens for x=a/2.

$$\begin{array}{rc} & a^2 + 4 \cdot a - 12 < 0 \\ \Leftrightarrow & (a-2) \cdot (a+6) < 0 \\ \Leftrightarrow & (a > -6 \wedge a < 2) \vee (a < -6 \wedge a > 2) \\ \Leftrightarrow & -6 < a \wedge a < 2 \end{array}$$

Find the values of a for which

$$-x^2 + a \cdot x + a - 3 < 0$$

holds for all x.

```
(-x^2)+a*x+a-3 < 0
a-3 < x^2-a*x
a-3 < (x-a/2)^2-a^2/4
a^2/4+a-3 < (x-a/2)^2
a^2+4*a-12 < 4*(x-a/2)^2
(a-2)*(a+6) < 4*(x-a/2)^2
"This is inequality is required to be true for all x; it must be true when the right hand side takes its minimum value.  This happens for x=a/2."
(a-2)*(a+6) < 0
(a-2 > 0 and a+6 < 0) or (a-2 < 0 and a+6 > 0)
(a > -6 and a < 2) or (a < -6 and a > 2)
(-6 < a and a < 2) or false
-6 < a and a < 2
```

Your last answer was interpreted as follows:

$$-x^2 + a \cdot x + a - 3 < 0$$

$\Leftrightarrow$ $$a - 3 < x^2 - a \cdot x$$

$\Leftrightarrow$ $$a - 3 < \left(x - \frac{a}{2}\right)^2 - \frac{a^2}{4}$$

$\Leftrightarrow$ $$\frac{a^2}{4} + a - 3 < \left(x - \frac{a}{2}\right)^2$$

$\Leftrightarrow$ $$a^2 + 4 \cdot a - 12 < 4 \cdot \left(x - \frac{a}{2}\right)^2$$

$\Leftrightarrow$ $$(a - 2) \cdot (a + 6) < 4 \cdot \left(x - \frac{a}{2}\right)^2$$

This is inequality is required to be true for all x; it must be true when the right hand side takes its minimum value. This happens for x=a/2.

$$(a - 2) \cdot (a + 6) < 0$$

$\Leftrightarrow$ $$(a - 2 > 0 \land a + 6 < 0) \lor (a - 2 < 0 \land a + 6 > 0)$$

$\Leftrightarrow$ $$(a > -6 \land a < 2) \lor (a < -6 \land a > 2)$$

$\Leftrightarrow$ $$(-6 < a \land a < 2) \lor \mathbf{false}$$

$\Leftrightarrow$ $$-6 < a \land a < 2$$

Algebra and RE

Even if you abandon calculation to CAS, you have to set up computational proofs!

Algebra and RE

Even if you abandon calculation to CAS, you have to set up computational proofs!
CAS challenge: get your CAS to rewrite $-x^2 + ax + a - 3 < 0$ as

$$(a-2)(a+6) < 4 \cdot \left(x - \frac{a}{2}\right)^2$$

Student interface

Lots of design decisions:

- Text area for input: freedom.

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Student interface

Lots of design decisions:

- Text area for input: freedom.
- Should students be expected to show logic?
- Should students indicate what they have done?
- (Semi-automatic assessment of proofs?)

Pell's *Algebra* 1668

74 — *Resolution of Problemes.*

Prob. IX. There be three Numbers in *continual* Proportion; their sum is 74, and the sum of their Squares 1924.

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To what extent can we change mathematics?

Pragmatists would say $\{\}$.

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- Use of natural domains?
- Cancelling and tracking side conditions.
- Multiplicities of roots.

Modified rules

(1) Multiplication does not retain equivalence.

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$$CA = CB \wedge C \neq 0 \Leftrightarrow A = B \wedge C \neq 0. \tag{2}$$

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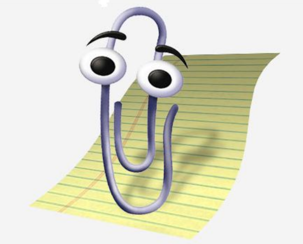
(Auditing)

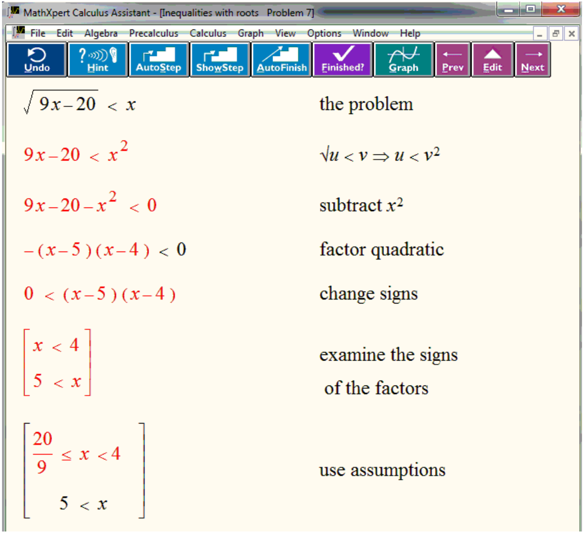
Design of algebra/software

Immediate feedback: assessment system $\rightarrow$ “algebra assistant”?

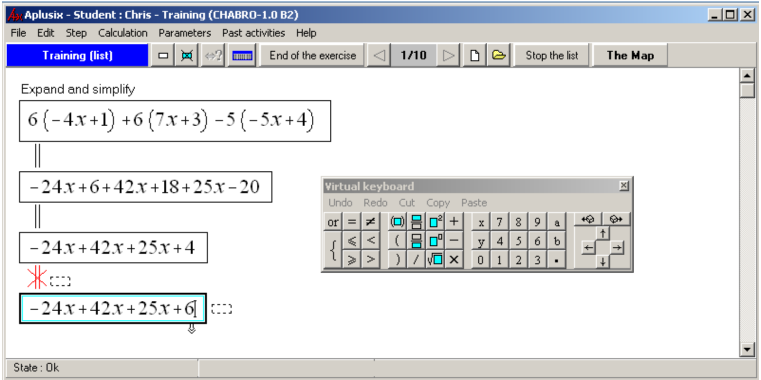
You appear to be implicitly enlarging the domain of x. Did you want some help with that?



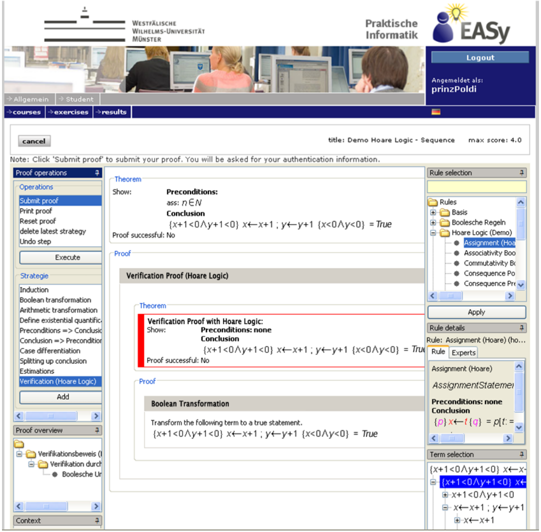
MathExpert


MathXpert Calculus Assistant - [Inequalities with roots Problem 7]
File Edit Algebra Precalculus Calculus Graph View Options Window Help
Undo Hint AutoStep ShowStep AutoFinish Finished? Graph Prev Edit Next
$\sqrt{9x-20} < x$
the problem
$9x-20 < x^2$
$\sqrt{u} < v \Rightarrow u < v^2$
$9x-20-x^2 < 0$
subtract x^2
$-(x-5)(x-4) < 0$
factor quadratic
$0 < (x-5)(x-4)$
change signs
$\left[\begin{array}{l} x < 4 \\ 5 < x \end{array}\right]$
examine the signs
of the factors
$\left[\begin{array}{l} \frac{20}{9} \le x < 4 \\ 5 < x \end{array}\right]$
use assumptions

Aplusix - reasoning by equivalence



EASy system


Westfälische Wilhelms-Universität Münster
Praktische Informatik
EASy
Logout
Angemeldet als:
prinzPoldi
Allgemein
Student
courses
exercises
results
cancel
title: Demo Hoare Logic - Sequence
max score: 4.0
Note: Click 'Submit proof' to submit your proof. You will be asked for your authentication information.
Proof operations
Operations
Submit proof
Print proof
Reset proof
delete latest strategy
Undo step
Execute
Strategie
Induction
Boolean transformation
Arithmetic transformation
Verification (Hoare Logic)
Add
Proof overview
Context
Theorem
Show:
Preconditions:
Conclusion
Proof successful: No
Proof
Verification Proof (Hoare Logic)
Verification Proof with Hoare Logic:
Preconditions: none
Boolean Transformation
Transform the following term to a true statement.
Rule selection
Rules
Basis
Boolesche Regeln
Hoare Logic (Demo)
Apply
Rule details
Rule
Experts
Assignment (Hoare)
Term selection

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- RE could be used to solve a wider range of problems than is currently the case.
- Personal opinion: we should pay more attention to them.

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- An opportunity to reflect on how algebra is taught...
- There are important other forms of reasoning beyond RE.