E-assessment at masters level: a case study from the Open University's MSc in Mathematics using STACK

Ben Mestel
School of Mathematics & Statistics
The Open University, UK

Joint work with Grahame Erskine & Tim Lowe

The Open University

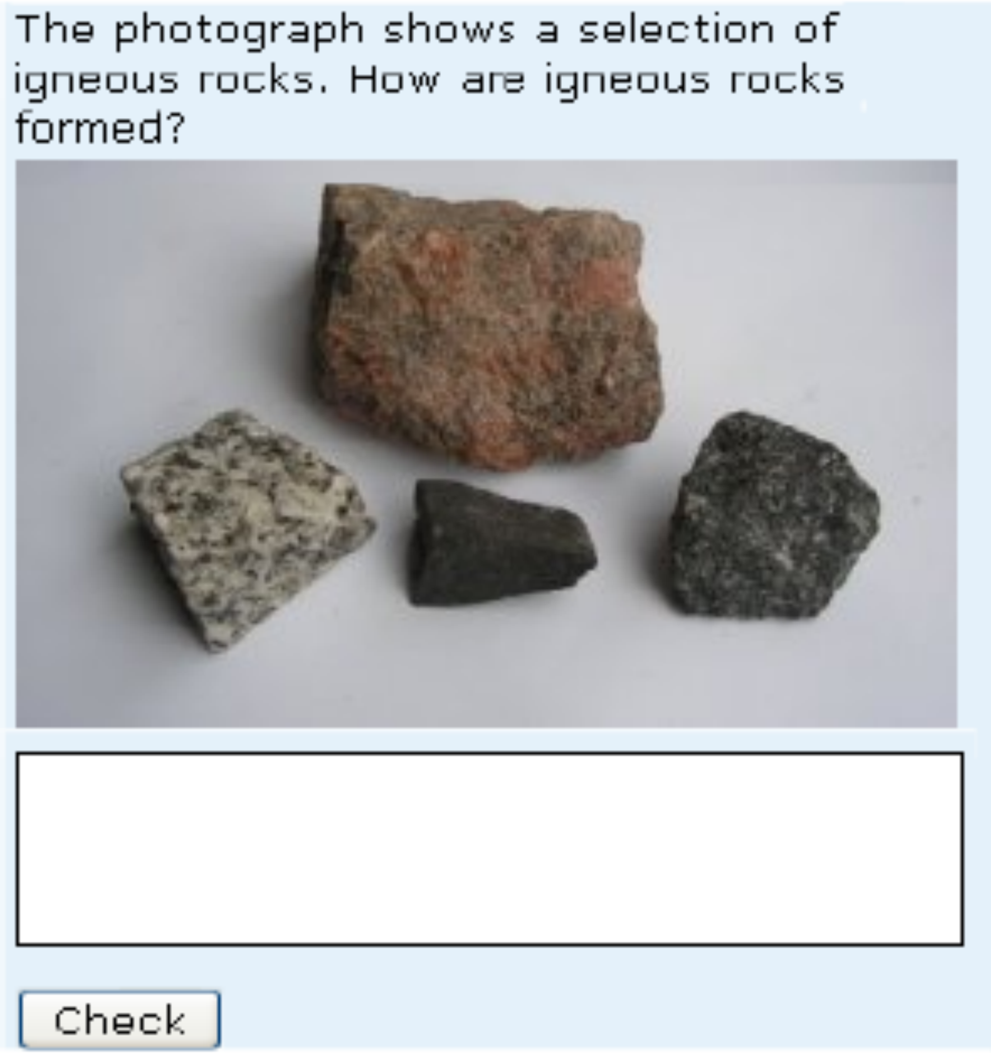
EAMS 2021
2 July 2021

The Open University (OU)

Berrill Building, Walton Hall Campus, Milton Keynes

- Britain's distance learning university since 1969
- Based in Milton Keynes but with offices around the UK in England, Northern Ireland, Scotland and Wales
- 168,000 students, most studying part-time
- Levels from opening to BSc to Masters and PhD
- *Modus Operandi:* high quality course notes, supplemented by tutor-marked assignments, correspondence tuition and face-to-face/online tutorials

e-Assessment at the OU



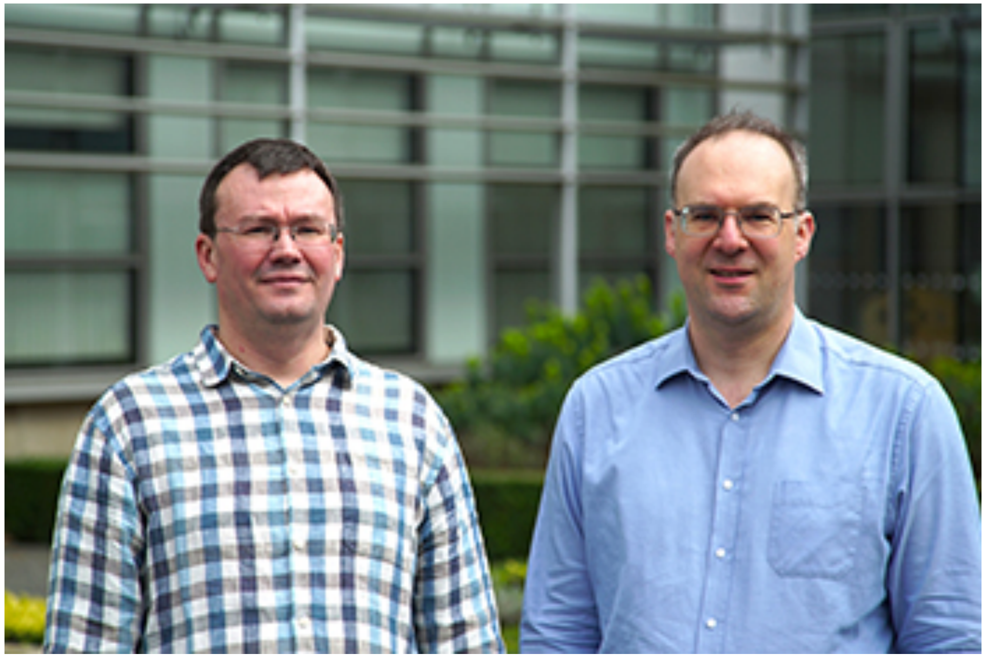
- The OU was a pioneer in computer-marked objective testing (i.e. multiple choice)
- Developed its own bespoke e-assessment system – OpenMark (excellent system but required professional programming in many instances)
- Since 2008 significant investment in the STACK system although and other systems used as well

Early e-assessment at the OU pioneered by Phil Butcher

STACK AT THE OU

- Integrated as part of the Moodle Quiz Engine in the Virtual Learning Environment
- 1.3 million STACK questions annually

Chris Sangwin



Tim Lowe and Tim Hunt

Mathematics Master of Science

Alan Turing Building at Open University Campus in Milton Keynes, Spring 2013
Photo by: [Chmee2]

- Mathematics MSc offered since 1980s
- About 500 students
- 2 - 6 years to complete 6 30-credit modules
- Modules cover a range of subjects in pure and applied mathematics but not statistics
- STACK used for two modules:
  - M820 Calculus of Variations and Advanced Calculus
  - M823 Analytic Number Theory I
- Modules mainly assessed by final exam

M820 Calculus of Variations & Advanced Calculus

- Introductory module of the MSc programme
- About 130 students annually
- 8 multi-question STACK quizzes were developed to assist with learning and preparation for the module exam (currently taken remotely)
- 30 STACK questions in total
- STACK questions developed first in 2017 and extended in 2020

8 multi-question STACK quizzes

- Solution of the **Euler-Lagrange differential equation** for quadratic functionals to obtain the stationary path.
- Solution of variational problems through the **first-integral,** for those functionals permitting such an approach.
- Local classification of stationary paths of functionals into minima, maxima and saddle points, through the analysis of the **Jacobi differential equation**.
- Calculation of the **Noether invariants** (first-integrals) for functionals invariant under a scale change in the variables.
- Diagonalisation of quadratic functionals involving **two dependent variables**, therefore allowing the stationary paths to be calculated by the solution of two independent variational problems.
- **Constrained variational problems** with integral constraints.
- Non-fixed endpoint conditions using the **transversality condition**.
- The use of the **Rayleigh-Ritz method** to find an upper bound for the least eigenvalue of a Sturm-Liouville problem.

Crash course in Calculus of Variations

- Applications in physics, biology, control engineering, economics and chemistry and more...
- Extremal paths of functionals of the form

$$\int_a^b dx\, F(x, y, y')$$

- (and generalisations) together with boundary conditions and constraints.
  - Stationary paths solve Euler-Lagrange equation

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$$

- Second order differential equation – boundary value problem

Classification via Jacobi equation

- Classification of stationary paths into local maxima, local minima and saddles – generally a hard problem ☹
- One approach – solve the Jacobi differential equation initial value problem
- $\frac{d}{dx}\left(P\frac{du}{dx}\right) - Q\,u = 0,\ u(a) = 0,\ u'(a) = 1$, where
- $P = \frac{\partial^2 F}{\partial y'^2}$, and $Q = \frac{\partial^2 F}{\partial y^2} - \frac{d}{dx}\left(\frac{\partial^2 F}{\partial y \partial y'}\right)$, evaluated on the stationary path $y(x)$.
- If $u(x)$ has no zeros ("conjugate points" in the jargon) in the half-open interval $(a, b]$ then $y(x)$ is a local minimum if $P > 0$ on $[a, b]$; a local maximum if $P < 0$ on $[a, b]$; and a saddle if P changes sign on $[a, b]$. Further analysis is needed in other cases.

Devising a question

- Restrict to quadratic functionals of the form

$$\int_a^b dx\,\left(\alpha_0\, x^m {y'}^2 + \beta_0\, x^{m-2} y^2\right),$$

  on $[a,b]$, with boundary conditions $y(a) = A,\ \ y(b) = B$, where A, B are constants, chosen to reduce the algebra

- The constants a, b and α_0, β_0 and m specialised as below.

- For this example the Euler-Lagrange equation and the Jacobi equation are the same equation (but different problems !)

Solving Jacobi

- Jacobi equation is Euler equation

$$A_2 x^2 \frac{d^2u}{dx^2} + A_1 x \frac{du}{dx} + A_0 u = 0,$$
$$u(a) = 0, \qquad u'(a) = 1,$$

- where $A_2 = \alpha_0$, $A_1 = m\alpha_0$, and $A_0 = -\beta_0$. The usual transformation $x = e^t$ leads to

$$A_2 \frac{d^2u}{dt^2} + (A_1 - A_2)\frac{du}{dt} + A_0 u = 0,$$
$$u(\log a) = 0, \qquad u'(\log a) = 1.$$

- Choose auxiliary $\lambda^2 - 2\rho\lambda + \rho^2 + \omega^2 = 0$, where ρ and ω are ‘nice’ constants with $\omega \geq 0$ and ρ, ω not both zero.

- $u(x) = \left(\frac{a}{\omega}\right)\left(\frac{x}{a}\right)^\rho \sin(\omega \log(x/a))$ for $\omega > 0$ and

$u(x) = a\left(\frac{x}{a}\right)^\rho \log(x/a)$ for $\omega = 0$.

Restricting and randomizing

- To reduce complexity, restrict to $a = 1$ and $\omega > 0$, so $u(x) = \frac{1}{\omega} x^{\rho} \sin(\omega \log x)$
- Zeros of $u(x)$ in $(1, b]$ if and only if $\omega \log b \geq \pi$
- Vary choice of ω and b to flip between the two cases
  - no zeros & Jacobi test applies
  - at lest one zero & Jacobi test fails.
  - In the first case,
    - minimum if $P(x) = 2\alpha_0 > 0$
    - maximum if $P(x) = 2\alpha_0 < 0$.
- (P is of fixed sign (for $\alpha_0 \neq 0$) so the stationary path is either a local maximum or a local minimum.)

Euler-Lagrange equation

- Euler-Lagrange boundary value problem:

$$A_2 x^2 \frac{d^2 y}{dx^2} + A_1 x \frac{dy}{dx} + A_0 y = 0,$$
$$y(a) = A, \qquad y(b) = B$$

- For simplicity choose $A = 0$ and $B = kb^\rho \sin \omega \log b$, where k is a small random positive integer
- Stationary path $y(x) = k\, x^\rho \sin(\omega \log x)$

STACK Jacobi Quiz

M820-17J Home | Assessment | Tutorials | Forums | Resources | News | Help ? | Search M820-17J

M820-17J Home > Resources > Online practice quizzes - to help with learning and revision > **Q5. Using Jacobi's equation**

Questions

1 2 3 4 5

Finish attempt ...

Question 1 Tries remaining: 3

Marked out of 1.00 | Flag question

This question provides practice in the classification of stationary paths using Jacobi's equation. In this first part you are asked to calculate the Euler-Lagrange equation of the functional.

Consider the functional

$$S[y] = \int_1^7 \mathrm{d}x \left(\frac{4\,y^2}{x^3} - \frac{2\,y'^2}{x} \right)$$

with boundary conditions $y(1) = 0,\ y(7) = 7\,\sin\left(\ln(7)\right)$.

Find the Euler-Lagrange equation for $S[y]$ in the form $G(x, y, y', y'') = 0$ where G is an expression involving x, y, y' and y''. Simplify your expression where possible.

(To input y' you should type **yp** in the input box and to input y'' you should type **ypp**. Use * and ^ to indicate multiplication and powers: for example xy'^2 is entered as **x*yp^2**.)

$G(x, y, y', y'') =$ [ ] $= 0$

Check

Next page >

Question in practice: feedback

Your answer is correct.

The Jacobi equation is, in this case, the same as the Euler-Lagrange equation. So the general solution of the Jacobi equation in terms of $t = \ln x$ is

$$u(t) = \alpha e^t \sin(t) + \beta e^t \sin(t - \ln(7)).$$

In terms of x, this becomes

$$u(x) = \alpha x \sin(\ln(x)) + \beta \sin\left(\ln\left(\frac{x}{7}\right)\right) x.$$

The initial condition $u(1) = 0$ gives $\beta = 0$.

So $u(x) = \alpha x \sin(\ln(x))$ and $u'(x) = \alpha(\sin(\ln(x)) + \cos(\ln(x)))$.

The condition $u'(1) = 1$ then gives $\alpha = 1$. So the solution of the Jacobi equation is

$$u(x) = x \sin(\ln(x)).$$

From the theory, in order to be able to use the Jacobi equation method to determine the nature of the stationary path, we require that there should be no point $\tilde{a}$ conjugate to $a = 1$ in the interval $1 < \tilde{a} \leq 7$. In other words, there should be no point $\tilde{a}$ in this interval for which $u(\tilde{a}) = 0$. If this holds, then the stationary path is a weak local minimum if $P(x) > 0$ everywhere on the interval $1 \leq x \leq 7$, and a weak local maximum if $P(x) < 0$ on the interval. In our case $P(x) = -\frac{4}{x}$ and so $P < 0$ on the interval.

Now $u(x) = 0$ if and only if $\ln(x) = \pi k$, or $x = e^{\pi k}$, for some integer k. Thus the smallest zero of u with $\tilde{a} > 1$ occurs at $\tilde{a} = e^{\pi}$.

Since the smallest value of $\tilde{a} > 1$ for which $u(\tilde{a}) = 0$ is at $\tilde{a} = e^{\pi} = 23.140\ldots > 7$, there is no point $\tilde{a}$ conjugate to 1 in the interval $1 < \tilde{a} \leq 7$.

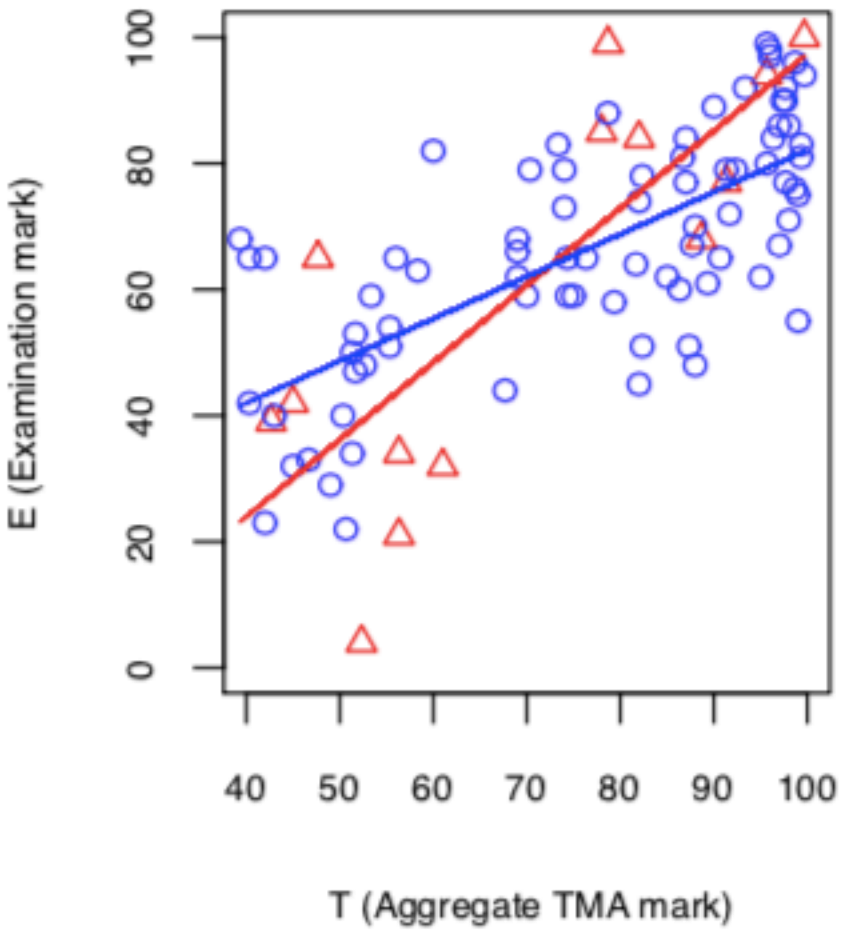
Since $P < 0$, it follows that the stationary path is a weak local maximum.

How did it STACK up?

- Well received by students
- “Started using these for exam revision. Great questions and the answers are very well written. Would love to see more types of questions.”
- “Practice quizzes were fabulous for revision”
- “In particular the practice quizzes and screencasts were by far the most helpful!”

Modern Book Printing, Walk of Ideas, Berlin, 2006. Photo: Matthew6476 CCSAL

Statistical analysis of first-year



Blue: interacted with quizzes

$E = 15.20 + 0.67T$

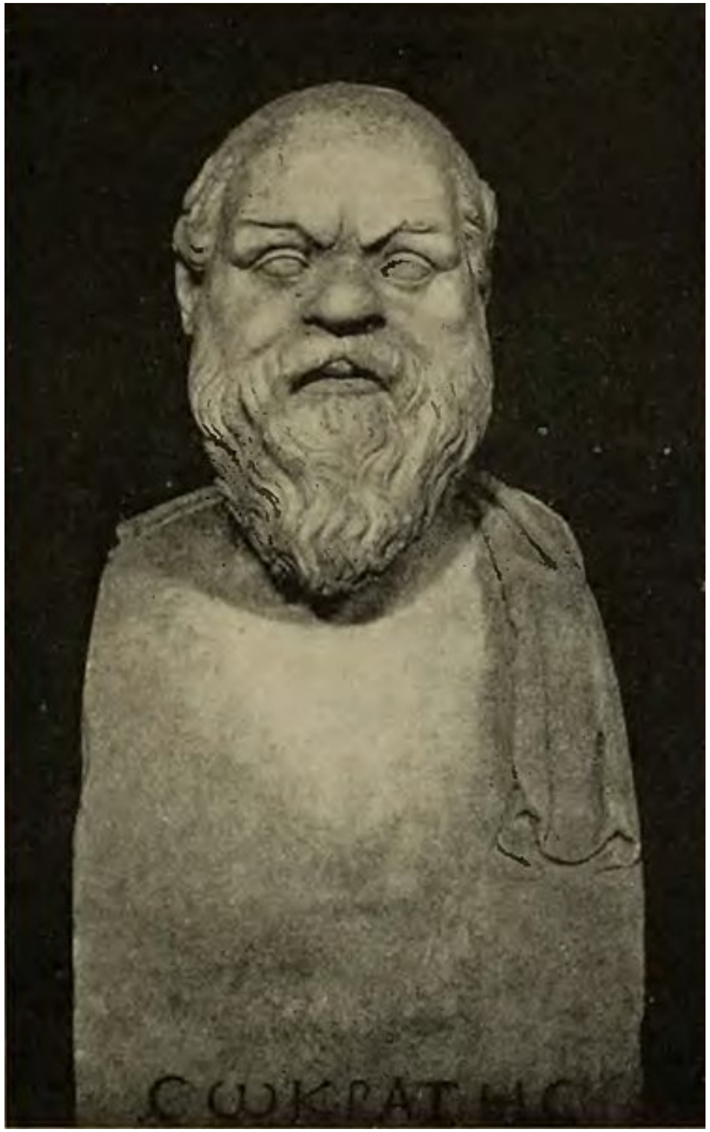
Red: did not interact with quizzes

$E = -24.99 + 1.22T$

Exam mark vs
continuous assessment mark

From (Mestel & Lowe, 2019)

Some (STACK) authoring tips



- Mostly common sense, but hopefully useful
- Not hard and fast rules, rather suggestions
- Based on our experience of authoring e-assessment questions in STACK and other systems
- Feel free to disagree !

Bust of Socrates. Photo: Bregi

Some question authoring tips

- Design and implementation: iterative process, useful to have more than one person involved
- Start from the solution and derive the question
- Check the answers - don’t pattern match
- Handle each case separately – don’t try to do everything in one go
- Helpful to solve the problem in CAS (Maxima) first, before implementation.
- Randomise in Maxima and then select a good set of parameter values
- Use multi-question quizzes not questions with many parts
- Hard work to coax Maxima to display formulae nicely
- Feedback is often the most time-consuming part and it is frequently omitted

2^{nd} order inhomogeneous linear ODE with constant coefficients

- $a_2 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0\, y = f(x) = \begin{cases} b_0 + b_1 x + \cdots + b_k x^k \\ b_0 \cos \mu x + b_1 \sin \mu x \\ b_0\, e^{\mu x} \end{cases}$
- $y(x_0) = y_0,\ \ y'(x_0) = z_0$
- Discriminant $\Delta = a_1^2 - 4a_2 a_0 \begin{cases} > 0 \\ < 0 \\ = 0 \end{cases}$, eigenvalue $\begin{cases} \lambda_1\,, \lambda_2 \\ \lambda \pm i\omega \\ \lambda, \lambda \end{cases}$
- $y = y_c + y_p,\ \ y_c = C_1\, e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$ (for $\Delta > 0$)
- $y_p = c_0\, e^{\mu x}\ \ (\mu \neq \lambda_1, \lambda_2)$

Thank you !

- Grahame Erskine and Ben Mestel, Developing STACK practice questions for the Mathematics Masters Programme at the OU, MSOR Connections, 17 (1) (2018), http://dx.doi.org/10.21100/msor.v17i1.896
- Ben Mestel and Tim Lowe, Using STACK to support student learning at masters level: a case study, Teaching Mathematics and its Applications, 03 (2019), https://dx.doi.org/10.1093/teamat/hrz001